TAKING ACCOUNT OF LOCAL SINGULARITIES OF A VORTEX LIFTING SURFACE IN THE METHOD OF DISCRETE VORTICES

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The method of discrete vortices (MDV) is an efficient method of solving singular integral equations encountered in wing theory. This explains its successful utilization to solve a broad class of linear and nonlinear problems of stationary, nonstationary, separation, and nonseparation flows around a wing. As a rule computation results are in good agreement with experimental data. Extensive literature (see [1-4], say) is devoted to the method of discrete vortices. Many MDV aspects have been subjected to a detailed analysis, including questions of the convergence of different computational schemes and expansion of its domain of application [5].

In the majority of papers a finite-span wing is modeled by a system of discrete horseshoe vortex filaments (Π-shaped vortices). For this the wing is divided into a finite number of elements, each of which is replaced by one Π-shaped vortex. The condition of fluid nonpenetration through the wing is satisfied at control points ordinarily chosen at equal distances between the discrete vortices. Such a computation scheme assures convergence of the approximate solution to the exact as the number of elements increases for any fixed internal part of the wing [5] but yields an uneliminable error for discrete vortices located at the wing leading edge (under conditions of nonseparation flow of this edge). An analogous situation holds also when modeling the wing by a system of closed vortex frames.

It is shown in [6, 7] that a change in the control point location while converving the discrete vortex position can result in convergence of the approximate to the exact solution on the whole wing including its edge. For this the control points should be chosen from the condition that a continuous vortex layer simulating the wing and the vortex sheet behind it will induce the same velocities at these points as does the system of discrete vortices. The problem of finding the control points can be solved separately for each part of the wing with local singularities of the vortex lifting surface that are given in conformity with the selected class of solutions of the initial singular integral equation taken into account.

Appropriate control points are determined in this paper for a finite span rectangular wing in cases of its simulation by Π-shaped vortices and closed vortex frames. Examples of a computation of stationary aerodynamic characteristics and apparent masses are presented that display the high efficiency of the computational schemes being proposed.

1. Let us first consider a scheme of Π-shaped vortices. To select the control points correctly in this scheme it is necessary to know the singularities of the vortex lifting surface simulating the wing.

Let a stationary ideal incompressible fluid flow around the ring and let the problem of the flow be solved within the framework of linear thin wing theory. Let us couple a Cartesian Oxyz coordinate system with the wing (Fig. 1). We represent the vector intensity of the vortex surface simulating the wing and the vortex sheet behind it in the form

$$\boldsymbol{\gamma}(x,\ z) = \gamma_x(x,\ z)\mathbf{x}^0 + \gamma_z(x,\ z)\mathbf{z}^0$$

($\mathbf{x}^0$, $\mathbf{z}^0$ are the coordinate axis directions). In linear theory γ_z is the intensity of the attached and γ_x of the free vortices. The functions γ_x, γ_z are connected by the equation $\operatorname{div}\boldsymbol{\gamma} = 0$. On the vortex sheet $\gamma_z = 0$ and $\partial\gamma_x/\partial x = 0$. The following conditions

$$\gamma_x(0,\ z) = 0 \text{ for } |z| \leqslant l,\ \gamma_z(x,\ -l) = \gamma_z(x,\ l) = 0 \text{ for } 0 \leqslant x \leqslant b$$

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should be satisfied on the wing leading and side edges for nonseparation flow around these edges.

We assume that the solution of the problem of the flow around the wing is sought in the class of functions γ_z bounded at the wing trailing edge and unbounded at the leading edge. Then $\gamma_z(b, z) = 0$ at trailing edge points (Zhukovskii postulate) and in the neighborhood of the leading edge ($x = 0$, $|z| \leq \ell$) the function γ_z has the singularity $\gamma_z \sim 1/\sqrt{x}$. As regards the functions γ_x, then in the neighborhood of the side edges ($0 < x < b$, $|z| = \ell$) $\gamma_x \sim 1/\sqrt{l^2 - z^2}$. Hence, at points of the wing ($0 \leq x \leq b$, $|z| \leq \ell$) the functions γ_x, γ_z are written thus

$$\gamma_x(x, z) = \sqrt{\frac{x}{l^2 - z^2}} R_1(x, z), \quad \gamma_z(x, z) = \sqrt{\frac{b-x}{x}(l^2 - z^2)} R_2(x, z), \tag{1.1}$$

where R_1, R_2 are functions having no singularities at the wing edges.

Now, let us partition each half-wing into a finite number of elements. To do this we take points $x_i = i\Delta_1$, $i = 1, \ldots, N_1$ at the root chord (the x axis) and the points $z_j = j\Delta_2$, $j = 1, \ldots, N_2$ ($\Delta_1 = b/N_1$, $\Delta_2 = \ell/N_2$) at the leading edge of the left half-wing (the z axis). Drawing lines parallel to the x and z axes through them, we obtain $N = N_1 \times N_2$ elements S_{ij}. We divide the right half-wing in an analogous manner.

Following the method of discrete vortices, we associate a Π-shaped vortex filament of constant intensity Γ_{ij} with each element S_{ij} by locating it at a distance $\mu\Delta_1$ from the leading edge of this element (see Fig. 1). The coefficient μ governing the location of the apex of the vortex filament on the element S_{ij} in fractions of its chord can be selected arbitrarily from zero to $\frac{1}{2}$ since for such μ the apex of a horseshoe vortex Γ_{ij} and the corresponding control point will not emerge beyond the limits of the element S_{ij}. The arbitrariness in the selection of μ is associated with the fact that the displacement of the whole system of Π-shaped vortices and control point on the wing while conserving the spacings between them does not affect the accuracy of the computation of the discrete vortex intensity Γ_{ij}. In the majority of computational schemes $\mu = \frac{1}{4}$ is given.

By the definition of a Π-shaped vortex

$$\Gamma_{ij} = \int\limits_{x_{i-1}}^{x_i} \gamma_z(x, z)\, dx, \quad z_{j-1} < z < z_j. \tag{1.2}$$

it follows from (1.2) that simulating a wing by a system of Π-shaped vortices corresponds to the assumption of the independence of the function γ_z from z in each element S_{ij}. At the same time (1.2) allows any dependence of γ_z on x in the strip $z_{j-1} < z < z_j$ including the singularity on the wing leading edge and zero at the trailing edge. Therefore, according to (1.1) and (1.2), the solution of the problem of flow around a wing by the method of Π-shaped vortices permits representation of the intensity of the attached vortices γ_z in the form

$$\gamma_z(x, z) = \sqrt{\frac{b-x}{x}} f_j(x), \quad z_{j-1} < z < z_j, \quad j = 1, \ldots, N_2. \tag{1.3}$$

Let us turn to setting up a connection between Γ_{ij} and the intensity of the free vortices γ_x. In the computatoinal scheme under consideration the free vortices are modeled by "whiskers" converging to the apices of the Π-shaped vortices. The vector intensities of the "whiskers" convergent with each attached vortex Γ_{ij} are equal in absolute value and opposite in direction. Consequently, their total intensity at the element S_{ij} equals zero. The sum of the "whiskers" intensities convergent to adjacent wing elements and located at the junction between them (at the sections $z = z_j$) differs from zero in the general sense. In this connection, the function γ_x should not be determined on S_{ij} but on the wing elements S_{ij}^* shifted a distance $\Delta_2/2$ along the z axis with respect to S_{ij} (see Fig. 1). Consequently, the elements $S_{iN_2}^*$ abutting on the wing side edge have the width $\Delta_2/2$ while the rest are of the width Δ_2 (the elements $S_{i0}{}^*$ are divided in halves by the root chord). The intensity of the free vortices γ_x is then determined on $S_{ij}{}^*$ in terms of the sum of the intensities of all the "whiskers" passing through $S_{ij}{}^*$. By analogy with (1.3), the function γ_x can here be written as

$$\gamma_x(x, z) = \frac{1}{\sqrt{l^2 - z^2}} g_i(z), \quad x_{i-1} < x < x_i, \quad i = 1, \ldots, N_1. \tag{1.4}$$

Let us turn to finding directly the location of the control points on the wing when it is simulated by Π-shaped vortices with the dependences (1.3) and (1.4) taken into account. Let us represent the coordinates of the control points x_{0i}, z_{0j} in the form

$$x_{0i} = \Delta_1(i - 1 + \nu_{xi}), \ i = 1, \ldots, N_1, \ z_{0j} = \Delta_2(j - \nu_{zj}), \quad j = 1, \ldots, N_2. \tag{1.5}$$

The coefficients ν_{xi}, ν_{zj} yield the dimensionless distance between a control point on S_{ij} and the edges of this element in fractions of Δ_1 and Δ_2, respectively. Following [6], we evaluate ν_{xi}, ν_{zj} separately for each small fixed domain within the wing and near the leading, trailing, and side edges independently of the influence of the remaining part of the vortical surface. To do this, we limit ourselves in each domain to the first approximation for γ_x, θ_z. For the interior domains such an approximation is γ_x = const, γ_z = const, for domains near the leading edge $\gamma_x = 0$, $\gamma_z = \text{const}/\sqrt{x}$, trailing edge γ_x = const, $\gamma_z = \text{const} \times \sqrt{b - x}$, and near the side edges $\gamma_x = \text{const}/\sqrt{l^2 - z^2}$, $\gamma_z = 0$.

We now divide each domain into elements of the form S_{ij}, $S_{ij}{}^*$, we replace the vortex layers on them by discrete vortices, we calculate the velocities induced by the system of discrete vortices and the continuous vortex layer, and we find the points at which these velocities are in agreement. By increasing the number of elements in the fixed domain under consideration to infinity, we consequently obtain the coefficients ν_{xi}, ν_{zj} for the limit case $N_1 \to \infty$, $N_2 \to \infty$. The ν_{xi}, ν_{zj}, calculated in such manner that govern the coordinates of the control points x_{0i}, z_{0j}, permit solution of the problem of the flow around a wing for an arbitrarily large number of discrete vortices. In a first approximation we obtain

$$\begin{gathered}\nu_{x1} = \mu + 0.55, \ \nu_{xi} = \mu + 0.5 \text{ for } i = 2, \ldots, N_1 - 1, \ \nu_{xN_1} = \mu + 0.38, \\ \nu_{zj} = 0.5 \text{ for } j = 1, \ldots, N_2 - 1, \ \nu_{zN_2} = 0{,}4.\end{gathered} \tag{1.6}$$

In other words, the standard scheme of a uniform arrangement of vortices and control points is conserved on the whole wing except for the elements abutting on the edges. But the control points are shifted somewhat backward near the leading edge, forward near the trailing edge, and closer to the edges at the side edges.

2. Let us clarify the influence of local singularities of the vortical lifting surface on the analysis of finite-span wing hydrodynamic characteristics by the method of Π-shaped discrete vortices. Let a stationary ideal incompressible fluid flow around the wing at an angle of attack α, and let the corresponding boundary value problem be solved in a linear formulation [1]. We perform the computation for two schemes, one of which is the standard scheme (1/4, 3/4) of a uniform arrangement of discrete vortices and control points while the other differs from the first just by a different control point distribution given by

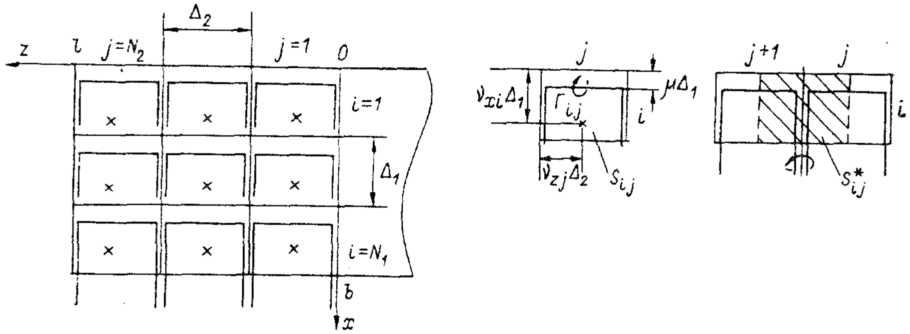

Fig. 1

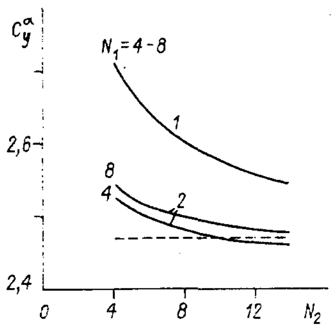

Fig. 2

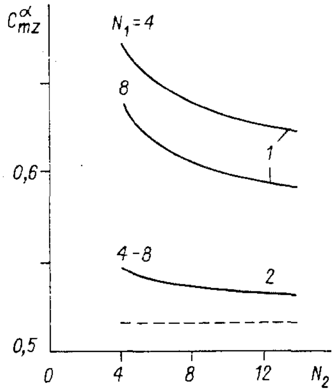

Fig. 3

(1.5), (1.6) near the wing edges. Strictly speaking, both computational schemes "are sensitive to" the unlimited growth in the vortex surface intensity during the approach to the wing leading edge, which is due to the arrangement of discrete vortices ahead of the control points in each element. But the second scheme permits taking account of the singularity at the leading edge more completely and, moreover, takes account of the nature of the behavior of the vortical intensity near the wing trailing and side edges.

The lift force Δp_{ij} acting on the element S_{ij} is: $\Delta p_{ij} = -\rho V \Gamma_{ij} \Delta_2$, where V is the flow velocity at infinity, and ρ is the liquid density. The total lift hence is

$$R_y = \frac{1}{2}\rho V^2 S C_y^\alpha \alpha = -2\rho V \Delta_2 \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \Gamma_{ij}.$$

Here S is the wing area; C_y^α is the derivative of the lift coefficient C_y at the angle of attack α.

The results of computing the coefficient C_y^α for a wing of span $\lambda = 2$ ($\lambda = 2\ell/b$) are represented in Fig. 2. The curves 1 correspond to the standard computation scheme, and 2 to the proposed scheme using (1.5) and (1.6) for $\mu = 1/4$. The dashed line is the result of a computation from [8] obtained by another method that is in good agreement with the data of other authors and can be taken as test values.

The materials presented show that a computation by the proposed scheme of selecting control points reduces to the test solution as the number of elements on the wings grows considerably more rapidly than by the standard scheme. This permits the required accuracy of the computations to be obtained for a smaller number of elements. Let us note that the lift force R_y determined by the standard scheme depends slightly on the number of partitions N_1 along the wing chord (in the limit case $\lambda \to \infty$ the coefficient is $C_y^\alpha = 2\pi$ for any N_1).

In addition to the lift force, the main characteristics of wing interaction with a stream are the moment of the hydrodynamic forces M_z relative to the z axis and the induced drag R_{xi}. Within the framework of linear thin wing theory

$$M_z = \int_{-l}^{+l} \left\{ \int_0^b x\gamma_z(x, z)\, dx \right\} dz, \quad R_{xi} = R_y \alpha - Q; \tag{2.1}$$

$$Q = -\pi\rho \lim_{x \to 0} \int_{-l}^{+l} x\gamma_z^2(x, z)\, dz \tag{2.2}$$

(Q is the suction force acting on the leading edge of a thin wing).

It is seen from (2.1) and (2.2) that the distribution of the vortex intensity $\gamma_z(x, z)$ over the wing surface must be known to evaluate M_z and R_{xi}. The discrete vortex method permits determination of just the total vortex intensity Γ_{ij} on the elements S_{ij}. Consequently, the problem occurs of an approximate determination of $\gamma_z(x, z)$ according to a given value of Γ_{ij}. It can be solved with a different degree of approximation. A spline approx-

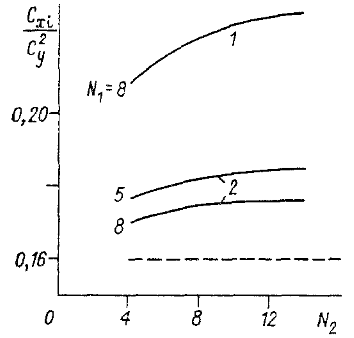

Fig. 4

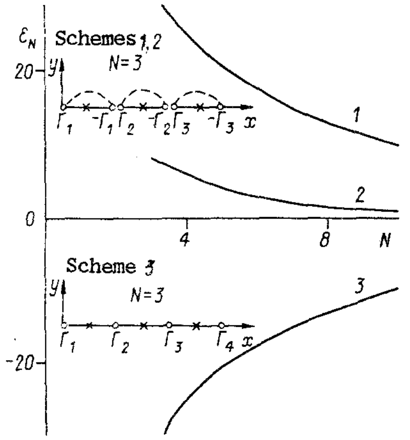

Fig. 5

imation of the function $\gamma_z(x, z)$ is used in this paper in each strip $z_{j-1} < z < z_j$, $j = 1, \ldots, N_2$, in the form

$$\gamma_z(x,z)=\begin{cases}(a_1^{(j)}+a_2^{(j)}\xi)/\sqrt{\xi}, & 0<\xi<\xi_1,\\ a_{3i-3}^{(j)}+a_{3i-2}^{(j)}(\xi-\xi_i)+a_{3i-1}^{(j)}(\xi-\xi_i)^2, & \xi_{i-1}\leqslant\xi\leqslant\xi_i,\\ & i=2,\ldots,N_1-1,\\ \sqrt{1-\xi}\left(a_{3N_1-3}^{(j)}+a_{3N_1-2}^{(j)}(1-\xi)\right), & \xi_{N_1-1}\leqslant\xi\leqslant 1\end{cases} \tag{2.3}$$

($\xi = x/b$, $\xi_i = i/N_1$).

The formulas (2.3) contain ($3N_1 - 2$) unknown coefficients. To find them we demand that the total vortex layer intensity according to (2.3) be equal to Γ_{ij} in each interval $[x_{i-1}, x_i]$ and the function γ_z and its derivative be integrable on the inner boundaries of these intervals. We consequently obtain a system of ($3N_1 - 2$) linear algebraic equations whose solution yields the desired coefficients.

The accuracy of the approximation (2.3) was estimated in the problem of plane stationary flow around a plate, which has the exact solution $\gamma_z(x) = -2V\alpha\sqrt{(b-x)/x}$. The discrete vortices $\Gamma_1, \ldots, \Gamma_{N_1}$, were determined from this solution and then the function $\gamma_z(x)$ was restored by using (2.3). Computations showed the high accuracy of the approximation (2.3). Thus, for a number $N_1 = 10$ of discrete vortices, say, the relative error in the approximation did not exceed 1% at all points of the plate, including the leading edge.

The approximation (2.3) was used to compute the moment of the hydrodynamic forces and the induced drag acting on the wing by means of (2.1) and (2.2). The quantities M_z and R_{xi} had the form

$$M_z=(1/2)\rho V^2SC_{mz}^{\alpha}\alpha,\quad R_{xi}=(1/2)\rho V^2SC_{xi}.$$

Results of computing the dimensionless coefficients C_{mz}^{α}, C_{xi}/C_y^2 for a wing of span $\lambda = 2$ are represented in Figs. 3 and 4 as obtained from the standard (curve 1) and proposed (2) schemes. The dashed lines correspond to the data in [8]. The results presented again display the high efficiency of the proposed computational scheme.

3. We now examine the scheme of closed vortex frames. Such a method of modeling the vortex surface is conveniently utilized to solve problems about the apparent masses of a vibrating wing [2]. In this case the utilization of closed vortex filaments of constant intensity permits automatic execution of the condition of no flow circulation around any wing section.

The solution for the carrier vortex surface intensity vector $\boldsymbol{\gamma}(x, z, t)$ in problems of apparent mass is sought in the class of functions for which the vector component $\boldsymbol{\gamma}$ tangential to the wing edges has a singularity at the edges. The influence of these singularities on the location of the control points is analogous to the influence of the Π-shaped vortex "whiskers." Namely, for wing elements abutting on the edges the control points should stand off 0.4 of an element length from the corresponding edge while located in their center in the remaining elements.

The efficiency of the proposed control point selection is estimated by the solution of the problem of the apparent masses of a plate of infinite span performing translational vibrations along the y axis. For a plane flow the vortex frames on a plate $y = 0$, $z \in [0, b]$ are degenerate into a system of discrete vortices Γ_1, $-\Gamma_1 + \Gamma_2$, ..., $-\Gamma_{N-1} + \Gamma_N$, $-\Gamma_N$ located at the points $x_0 = 0$, $x_1 = b/N$, ..., $x_{N-1} = (N - 1)b/N$, $x_N = b$, respectively. Shown as an example in Fig. 5 is the arrangement of the vortices and control points on a plate ($\lambda = \infty$) for the number of elements N = 3 (schemes 1 and 2), Scheme 1 corresponds to the standard control point arrangement between vortices [$x_{0k} = b(k - 0.5)/N$, $k = 1, ..., N$] while the control points in the scheme 2 are chosen with the local singularities of the function $\gamma_z(x, t)$ taken into account near the edges [$x_{01} = 0.4b/N$, $x_{0k} = b(k - 0.5)/N$ for $k = 2, ..., N - 1$, $x_{0N} = b(n - 0.4)/N$].

The computational scheme 3 that is obtained from scheme 1 by replacing each pair of vortices $(-\Gamma_{k-1} + \Gamma_k)$ by one vortex [2] is ordinarily used at this time to solve apparent mass problems. Consequently, we have a system of discrete vortices Γ_1, ..., Γ_{N+1} located at the ends of the plate elements while we select the control points at different spacings between the vortices (see Fig. 5). Strictly speaking, the lifting vortex surface in scheme 3 is not simulated by vortex frames but by Π-shaped vortices whose number is one greater than the number of control points. Consequently, in contrast to schemes 1 and 2, scheme 3 does not automatically satisfy the condition of circulation-free flow around a plate and it must be required for its satisfaction that the sum of the discrete vortices equal zero.

Within the framework of the linear thin wing theory, the hydrodynamic pressure at points of a plate vibrating in a fluid at rest (at infinity) without the formation of vortex wakes is determined by the Cauchy-Lagrange integral $p - p_\infty = -\rho\partial\varphi/\partial t$ (φ is the velocity potential and p_∞ is the pressure at an infinitely remote point). Hence, the projection of the total hydrodynamic force on the axis is

$$R_y = \rho \frac{\partial}{\partial t} \int_0^b \left\{ \int_0^x \gamma_z(u, t)\, du \right\} dx. \tag{3.1}$$

In the case under consideration of translational plate vibrations $R_y = -\lambda_y y$, where λ_y is the apparent mass coefficient: $y(t)$ is the plate displacement along the y axis. For an exact solution of the problem $\lambda_y = \rho\pi b^2/4$.

It follows from (3.1) that a connection must be established between the intensity of the initial vortex layer $\gamma_z(x, t)$ and the system of discrete vortices for a solution of the same problem by the discrete vortices method. This connection is set up by using a spline approximation of the function γ_z analogously to (2.3). Here γ_z should have a singularity at both edges of the plate. Moreover, it should be taken into account that the vortices Γ_1 and $-\Gamma_N$ determine the total vortex layer intensity in elements of length $\Delta_1/2$, while the remaining vortices govern elements of length Δ_1. As a result, $\gamma_z(x, t)$ is approximated by functions of the form

$$\gamma_z(x, t) = \begin{cases} (b_1 + b_2\xi)/\sqrt{\xi}, & 0 < \xi \leqslant \xi_1^*, \\ b_{3i-3} + b_{3i-2}(\xi - \xi_i^*) + b_{3i-1}(\xi - \xi_i^*)^2, & \xi_{i-1}^* \leqslant \xi \leqslant \xi_i^*, \\ & i = 2, \ldots, N, \\ (b_{3N} + b_{3N+1}(1 - \xi)/\sqrt{1 - \xi}, & \xi_N^* \leqslant \xi < 1 \end{cases} \tag{3.2}$$

[$\xi_k^* = x_k^*/b = (k - 0.5)/N$, $k = 1, ..., N$].

The formulas (3.2) contain 3N + 1 unknown coefficients b_1^*, ..., b_{3N+1}^* that are found from the N + 1 conditions

$$\Gamma_1 = \int_0^{x_1^*} \gamma_z\, dz;\ \Gamma_k - \Gamma_{k-1} = \int_{x_{k-1}^*}^{x_k^*} \gamma_z\, dx,\ k = 2, \ldots, N;\ -\Gamma_N = \int_{-x_N^*}^{b} \gamma_z\, dx \tag{3.3}$$

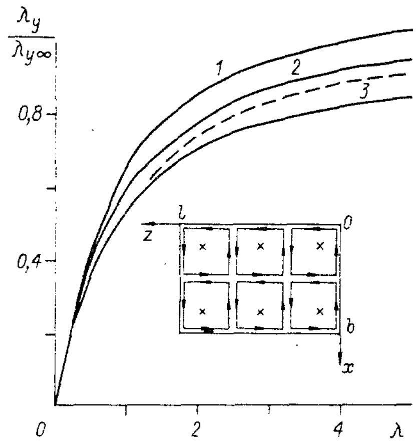

Fig. 6

and the 2N-continuity conditions of the function γ_z and its first derivative at the points x_1^*, ..., x_N^*. The quantities Γ_1, ..., Γ_{N+1} should be on the left in (3.3) when computing discrete vortices according to scheme 3.

The dependence of the relative error in computing the apparent mass coefficient λ_y according to schemes 1-3 on the number N of elements on the plate is represented in Fig. 5. The results presented show that a computation using scheme 2 that takes account of the local vortex layer singularities, will converge to the exact value λ_y much more rapidly than a computation using the standard schemes 1 and 3. It is interesting to note that schemes 1 and 3 yield identical relative errors in absolute value but of opposite sign in the test problem under consideration.

The apparent masses of a rectangular plate ($0 \le x \le b$, $y = 0$, $0 \le z \le \ell$) vibrating translationally along the y axis were computed by schemes analogous to 1-3. The plate was partitioned into $N_1 \times N_2$ elements, each of which was replaced by a closed vortex frame (an example of such a partition is presented in Fig. 6 for $N_1 = N_2 = 3$). The frames are replaced by Π-shaped vortices in elements abutting the trailing edge in scheme 3.

The results of computing the coefficient λ_y by using schemes 1-3 are represented in Fig. 6 as a function of the plate span $\lambda = \ell/b$ for $N_1 = N_2 = 10$. The quantity $\lambda_{y\infty}$ corresponds to the exact value of λ_y for $\lambda = \infty$. The dashed line is the experimental dependence of λ_y on λ from [9].

The data of a computation using scheme 2 are in good agreement with experiment for all values of λ while a computation using scheme 1 yields exaggerated and, using scheme 3, reduced results especially for large plate spans. Figs. 5 and 6 illustrate the high efficiency of the computation scheme 2.

Therefore, when modeling a wing by systems of Π-shaped vortices and closed vortex fames, the computational schemes in which the control points are selected with local singularities taken into account will assure high accuracy of the computation of the distributed and total hydrodynamic wing characteristics.

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